

# SPECTRAHEDRAL CONTAINMENT AND OPERATOR SYSTEMS WITH FINITE-DIMENSIONAL REALIZATION

TOBIAS FRITZ, TIM NETZER, AND ANDREAS THOM

**ABSTRACT.** We investigate when an abstract operator system has a finite-dimensional concrete realization. We prove a criterion involving the boundary of the system, and apply it to operator systems associated to a convex cone. The maximal operator system generated by a convex cone admits a finite-dimensional realization if and only if the cone is polyhedral. Among polyhedral cones, the minimal operator system has a finite-dimensional realization if and only if the cone is a simplex. These results are closely related to deciding inclusion of (free) spectrahedra. Our results imply that the present semidefinite algorithms for deciding inclusion are exact only for simplices, and they easily provide error estimates in other cases.

## 1. INTRODUCTION AND PRELIMINARIES

The results in this paper can be looked at from two sides. On the one hand, we examine abstract operator systems, and ask whether they admit a finite-dimensional concrete realization, i.e. a realization by matrices. This is an interesting and hard problem, which often involves determining the boundary representations of the system (see for example [1, 2, 8]). On the other hand, we examine spectrahedra, and in particular the problem of testing inclusion of such. Spectrahedra are the feasible sets of semidefinite programming, and have attracted a lot of attention in recent years, both from an applied and pure perspective (see [4] for an overview). A first idea for how to check inclusion via semidefinite programming stems from [3], and it has been discovered [13] that the method can only be fully understood by adding matricial levels to the spectrahedra, i.e. by examining their free versions. This idea has been further pursued in [7, 12, 17–19]. The connection between the two perspectives becomes clear by observing that free spectrahedra are essentially the same as operator systems with finite-dimensional realizations.

In this paper, we start with an abstract operator system and characterize when it admits a finite-dimensional realization (Theorem 2.3). We then investigate operator systems constructed from convex cones at scalar level, namely the minimal and the maximal operator system of a cone. We show that the maximal system admits a finite-dimensional realization if and only if the cone is polyhedral (Theorem 3.2), and the minimal system of a polyhedral cone is finite-dimensional realizable if and only if the cone is a simplex (Theorem 4.7). The minimal system of a non-polyhedral cone can also be finite-dimensional realizable (Example 4.10), but this seems to happen very rarely. Translated into the problem of testing inclusion of spectrahedra,

---

We thank Kai Kellner and Markus Schweighofer for interesting discussions, and Bill Helton, Igor Klep and Scott McCullough for helpful feedback on a draft.

Theorem 4.7 says the following. When checking inclusion of a fixed polytope in an arbitrary spectrahedron, the relaxation first introduced in [3] is exact for any spectrahedron if and only if the polytope is a simplex (Corollary 5.3). This is true independently of the representation of the polytope. We then give an easy proof of the existence of scaling factors for inclusion from [8, 19], and prove novel bounds for general spectrahedra.

Let us introduce the basic concepts. Throughout,  $\mathcal{V}$  denotes a  $\mathbb{C}$ -vector space with involution  $*$ , and  $\mathcal{V}_h$  is the  $\mathbb{R}$ -subspace of Hermitian elements. For any  $s \geq 1$ , the vector space  $\mathbb{M}_s(\mathcal{V}) := \mathcal{V} \otimes_{\mathbb{C}} \mathbb{M}_s(\mathbb{C})$  of  $s \times s$ -matrices with entries from  $\mathcal{V}$  comes equipped with the canonical involution defined by  $(v_{ij})_{i,j}^* := (v_{ji}^*)_{i,j}$ .

**Definition 1.1.** An *abstract operator system*  $\mathcal{C}$  on  $\mathcal{V}$  consists of a closed and salient convex cone  $\mathcal{C}_s \subseteq \mathbb{M}_s(\mathcal{V})_h$  for each  $s \geq 1$ , such that

- $A \in \mathcal{C}_s, V \in \mathbb{M}_{s,t}(\mathbb{C}) \Rightarrow V^*AV \in \mathcal{C}_t$ ,
- there is  $u \in \mathcal{C}_1 \subseteq \mathcal{V}_h$  such that  $u \otimes I_s$  is an order unit (or equivalently interior point) of  $\mathcal{C}_s$  for all  $s \geq 1$ .

*Remark 1.2.* (a) The topology in which each  $\mathcal{C}_s$  is required to be closed is understood to be the finest locally convex topology on  $\mathcal{V}$ .

(b) We usually consider the order unit  $u \in \mathcal{C}_1$  to be part of the structure of an operator system (as opposed to a mere property), which means that maps of operator systems are typically required to preserve it.

(c)  $(u \otimes I_s) \in \mathcal{C}_s$  is an order unit for all  $s$  if and only if this holds for  $s = 1$ . To show this, we start with an arbitrary element  $A = \sum_{i=1}^n v^{(i)} \otimes M_i \in \mathbb{M}_s(\mathcal{V})_h$  with  $v^{(i)} \in \mathcal{V}$  and  $M_i \in \mathbb{M}_s(\mathbb{C})$ . Assuming that  $u \in \mathcal{C}_1$  is an order unit, choose  $\lambda \in \mathbb{R}$  such that  $\pm v^{(i)} + \lambda u \in \mathcal{C}_1$ , and write  $M_i = P_i - Q_i$  as a difference of two positive semidefinite matrices. Then

$$\sum_i (v^{(i)} + \lambda u) \otimes P_i + (-v^{(i)} + \lambda u) \otimes Q_i = \sum_i v^{(i)} \otimes M_i + \lambda u \otimes \sum_i (P_i + Q_i)$$

is also in  $\mathcal{C}_s$ . Thus if  $\gamma \geq 0$  is large enough to ensure  $\gamma I_s - \sum_i (P_i + Q_i) \geq 0$ , then  $A + \gamma \lambda (u \otimes I_s) \in \mathcal{C}_s$ . So  $u \otimes I_s$  is indeed an order unit for  $\mathcal{C}_s$ .

By the Choi–Effros Theorem ([6], see also [22, Chapter 13]), for any abstract operator system  $\mathcal{C}$  there is a Hilbert space  $\mathcal{H}$  and a  $*$ -linear mapping  $\varphi: \mathcal{V} \rightarrow \mathbb{B}(\mathcal{H})$  with  $\varphi(u) = \text{id}_{\mathcal{H}}$ , such that for all  $s \geq 1$  and  $A \in \mathcal{C}_s$ ,

$$A \in \mathcal{C}_s \Leftrightarrow (\varphi \otimes \text{id})(A) \geq 0.$$

On the right-hand side, we use the canonical identification

$$\mathbb{M}_s(\mathbb{B}(\mathcal{H})) = \mathbb{B}(\mathcal{H}) \otimes_{\mathbb{C}} \mathbb{M}_s(\mathbb{C}) = \mathbb{B}(\mathcal{H}^s)$$

to define positivity of the operator. Such a mapping  $\varphi$  is called a *concrete realization* or just *realization* of the operator system  $\mathcal{C}$ . A realization  $\varphi$  is necessarily injective, since  $\mathcal{C}_1$  does not contain a nontrivial subspace.

**Definition 1.3.** For  $r \in \mathbb{N}$ , an abstract operator system  $\mathcal{C}$  is  *$r$ -dimensional realizable* if there is a realization with  $\dim \mathcal{H} = r$ . It is *finite-dimensional realizable* if it is  $r$ -dimensional realizable for some  $r \in \mathbb{N}$ .

Now assume that  $\mathcal{V}$  is finite-dimensional. After a suitable choice of basis, we can assume  $\mathcal{V} = \mathbb{C}^d$  with the canonical involution, and thus  $\mathcal{V}_h = \mathbb{R}^d$ . Then

$$\mathbb{M}_s(\mathcal{V}) = \mathcal{V} \otimes_{\mathbb{C}} \mathbb{M}_s(\mathbb{C}) = \mathbb{M}_s(\mathbb{C})^d, \quad \mathbb{M}_s(\mathcal{V})_h = \text{Her}_s(\mathbb{C})^d,$$

and a realization of  $\mathcal{C}$  just consists of self-adjoint operators  $T_1, \dots, T_d \in \mathbb{B}(\mathcal{H})_h$  with  $u_1 T_1 + \dots + u_d T_d = \text{id}_{\mathcal{H}}$  and

$$(A_1, \dots, A_d) \in \mathcal{C}_s \Leftrightarrow T_1 \otimes A_1 + \dots + T_d \otimes A_d \geq 0.$$

Finite-dimensional realizability then means that the  $T_i$  can be taken to be matrices.

**Definition 1.4.** A (classical) *spectrahedral cone* is a set of the form

$$\left\{ a \in \mathbb{R}^d \mid a_1 M_1 + \dots + a_d M_d \geq 0 \right\},$$

where  $M_1, \dots, M_d \in \text{Her}_r(\mathbb{C})$  are nonzero Hermitian matrices, and  $\geq 0$  again denotes positive semidefiniteness. For any  $s \geq 1$ , we define

$$\mathcal{S}_s(M_1, \dots, M_d) := \left\{ (A_1, \dots, A_d) \in \text{Her}_s(\mathbb{C})^d \mid M_1 \otimes A_1 + \dots + M_d \otimes A_d \geq 0 \right\}.$$

The family of cones  $\mathcal{S}(M_1, \dots, M_d) = (\mathcal{S}_s(M_1, \dots, M_d))_{s \geq 1}$  is called the *free spectrahedron* defined by  $M_1, \dots, M_d$ .

*Remark 1.5.* In order for a free spectrahedron to be an operator system, the positive cones must be salient and have an order unit. The first is equivalent to the  $M_i$  being linearly independent, and the latter happens in particular if there is  $u \in \mathbb{R}^d$  with  $\sum_i u_i M_i = I_r$ , in which case we take this  $u$  to be the order unit.

Classical spectrahedra are the feasible sets of semidefinite programming, which allows for efficient numerical algorithms (see for example [24, 25]). They share many properties of polytopes, which form a strict subclass. It is generally hard to decide whether a cone is spectrahedral, and a lot of recent research deals with questions arising in this area (see [4] for an overview). For example, the *inclusion problem* in its basic form asks whether

$$\mathcal{S}_1(M_1, \dots, M_d) \subseteq \mathcal{S}_1(N_1, \dots, N_d)$$

holds for given families of matrices  $M_i$  and  $N_j$ . In Section 5 we will explain how this problem relates to our results. For the moment, just note that a free spectrahedron with the properties of Remark 1.5 is (up to isomorphism) the same as a finite-dimensional realizable operator system.

## 2. A CRITERION FOR FINITE-DIMENSIONAL REALIZATIONS

In this section, we prove a criterion for operator systems to admit a finite-dimensional realization, namely Theorem 2.3 below. Throughout, let  $\mathcal{C} = (\mathcal{C}_s)_{s \geq 1}$  be an operator system on  $\mathcal{V} = \mathbb{C}^d$  with order unit  $u \in \mathcal{C}_1$ . Let  $\mathcal{C}_s^\vee$  denote the dual cone of  $\mathcal{C}_s$ , i.e. the set of all  $*$ -linear functionals on  $\mathbb{M}_s(\mathbb{C})^d$  that are nonnegative on  $\mathcal{C}_s$ . We begin by reviewing the separation method of Effros and Winkler.

**Lemma 2.1** ([9]). *Let  $\varphi \in \mathcal{C}_r^\vee$  be such that  $\varphi(u \otimes vv^*) > 0$  for all  $0 \neq v \in \mathbb{C}^r$ . Then there are  $M_1, \dots, M_d \in \text{Her}_r(\mathbb{C})$  with  $\sum_i u_i M_i = I_r$ , which generate a free spectrahedron containing  $\mathcal{C}$ , and such that:*

- (a) If  $A \in \text{Her}_s(\mathbb{C})^d$  is such that  $\varphi(V^*AV) = 0$  for some  $V \neq 0$ , then  $A$  is in the boundary of this free spectrahedron.
- (b) If  $A \in \text{Her}_r(\mathbb{C})^d$  is such that  $\varphi(A) < 0$ , then  $A$  is not in this free spectrahedron.

*Proof.* Let the  $N_1, \dots, N_d \in \text{Her}_r(\mathbb{C})$  be such that  $\varphi(B_1, \dots, B_d) = \sum_i \text{tr}(\overline{N}_i B_i)$  for all  $B \in \mathbb{M}_r(\mathbb{C})^d$ . The positivity assumption guarantees that  $\sum_i u_i N_i \geq 0$ . Even better, the assumption  $\varphi(u \otimes vv^*) > 0$  for all  $0 \neq v \in \mathbb{C}^r$  implies that  $\hat{N} := \sum_i u_i N_i > 0$ , and thus we can put  $M_i := \hat{N}^{-1/2} N_i \hat{N}^{-1/2}$  and have  $\sum_i u_i M_i = I_r$  by construction.

To show that the resulting free spectrahedron contains  $\mathcal{C}$ , consider  $A \in \mathcal{C}_s$ . Then for  $x = \sum_{j=1}^r e_j \otimes v_j$  with  $v_1, \dots, v_r \in \mathbb{C}^s$  and  $e_1, \dots, e_r$  the standard basis of  $\mathbb{C}^r$ , we have

$$(1) \quad \left\langle x, \left( \sum_i N_i \otimes A_i \right) x \right\rangle = \sum_i \text{tr}(\overline{N}_i V^* A_i V) = \varphi(V^*AV) \geq 0,$$

where  $V$  is the matrix with  $v_1, \dots, v_r$  as its columns. Therefore  $\sum_i N_i \otimes A_i \geq 0$ , which also implies  $\sum_i M_i \otimes A_i \geq 0$ , as was to be shown.

If  $\varphi(V^*AV) = 0$  for some  $V \neq 0$ , then  $A$  lies in the boundary of the free spectrahedron, since (1) shows that  $\sum_i N_i \otimes A_i$  and hence also  $\sum_i M_i \otimes A_i$  is not positive definite, resulting in (a). Part (b) works similarly.  $\square$

**Definition 2.2.** The *essential boundary* of  $\mathcal{C}$  is:

$$\partial^{\text{ess}} \mathcal{C}_s := \{A \in \mathcal{C}_s \mid \exists \varphi \in \mathcal{C}_s^\vee, \varphi(u \otimes vv^*) > 0 \text{ for all } v \in \mathbb{C}^s \setminus \{0\}, \varphi(A) = 0\}.$$

So an element is in the essential boundary if its minimal exposed face does not contain an element  $u \otimes vv^*$  with  $v \neq 0$ .

Example 4.2 showcases what the essential boundary of a particular operator system may look like.

**Theorem 2.3.** *An operator system is  $r$ -dimensional realizable if and only if it has the following property: for any  $n, s_1, \dots, s_n \in \mathbb{N}$  and  $A^{(i)} \in \partial \mathcal{C}_{s_i}$ , there exist  $0 \neq V_i \in \mathbb{M}_{s_i, r}(\mathbb{C})$  with*

$$\sum_{i=1}^n V_i^* A^{(i)} V_i \in \partial^{\text{ess}} \mathcal{C}_r.$$

*Proof.* First assume that the system is  $r$ -dimensional realizable, with defining matrices  $M_1, \dots, M_d \in \text{Her}_r(\mathbb{C})$ . For  $A^{(i)} = (A_1^{(i)}, \dots, A_d^{(i)}) \in \partial \mathcal{C}_{s_i}$ , there exist vectors  $v_1^{(i)}, \dots, v_r^{(i)} \in \mathbb{C}^{s_i}$  such that

$$x^{(i)} := \sum_{k=1}^r e_k \otimes v_k^{(i)} \neq 0,$$

and

$$\left( \sum_{j=1}^d M_j \otimes A_j^{(i)} \right) x^{(i)} = 0.$$

Let  $V_i$  be the matrix with columns  $v_1^{(i)}, \dots, v_r^{(i)}$ . Then  $V_i \neq 0$ , and some calculation shows that

$$\operatorname{tr} \left( \sum_{j=1}^d \overline{M}_j \sum_{i=1}^n V_i^* A_j^{(i)} V_i \right) = \sum_{i=1}^n \left\langle x^{(i)}, \left( \sum_{j=1}^d M_j \otimes A_j^{(i)} \right) x^{(i)} \right\rangle = 0.$$

This proves that  $\sum_i V_i^* A^{(i)} V_i \in \partial^{\text{ess}} \mathcal{C}_r$ , since the positive functional  $B \mapsto \operatorname{tr}(\sum_j \overline{M}_j B_j)$  is strictly positive on each  $u \otimes vv^*$  with  $v \neq 0$ .

For the converse direction, we use one of the key arguments from [14]. Let  $A^{(i)} \in \partial \mathcal{C}_{s_i}$  for  $i = 1, \dots, n$  be elements of the boundary. Then the assumption guarantees that there are  $V_i \neq 0$  with  $\sum_i V_i^* A^{(i)} V_i \in \partial^{\text{ess}} \mathcal{C}_r$ . This means that there is  $\varphi \in \mathcal{C}_r^\vee$  with  $\varphi(u \otimes vv^*) > 0$  for all  $v \in \mathbb{C}^s \setminus \{0\}$  and  $\varphi(\sum_i V_i^* A^{(i)} V_i) = 0$ . Since  $\varphi \in \mathcal{C}_r^\vee$ , this means that  $\varphi(V_i^* A^{(i)} V_i) = 0$  for each  $i$  separately. Hence Lemma 2.1 constructs matrices in  $\operatorname{Her}_r(\mathbb{C})$  which generate a free spectrahedron containing  $\mathcal{C}$ , and such that the  $A^{(i)}$  are in its boundary.

The existence of the order unit implies that the defining matrices of such a free spectrahedron are uniformly bounded. Therefore the tuples of matrices that define free spectrahedra containing  $\mathcal{C}$  form a compact set. We now choose a sequence of boundary elements  $A^{(i)} \in \mathcal{C}_{s_i}$  that are dense in the boundary at all matrix levels, and consider the sequence of free spectrahedra associated to all finite initial subsequences. By the compactness, this sequences of free spectrahedra containing  $\mathcal{C}$  must have an accumulation point. The free spectrahedron described by such an accumulation point again contains  $\mathcal{C}$ , and every  $A^{(i)}$  is in its boundary. We therefore have an  $r$ -dimensional realizable system with the same boundary as  $\mathcal{C}$ , and thus coincides with  $\mathcal{C}$ .  $\square$

We will see in Section 4 how this result can be used to show that certain operator systems are not finite-dimensional realizable.

### 3. THE MAXIMAL OPERATOR SYSTEM OF A CONE

In this and the next section, we start with a closed salient cone  $C \subseteq \mathbb{R}^d$  with order unit  $u$  and consider operator systems  $(C_s)_{s \geq 1}$  with  $C_1 = C$ . It is not hard to see that there is always a minimal and a maximal one. We start with the maximal system:

$$C_s^{\max} := \left\{ (A_1, \dots, A_d) \in \operatorname{Her}_s(\mathbb{C})^d \mid \forall v \in \mathbb{C}^s \ (v^* A_1 v, \dots, v^* A_d v) \in C \right\}.$$

We also write  $C^{\max}$  as shorthand for the family  $(C_s^{\max})_{s \geq 1}$ . This system is maximal in the sense that for any operator system  $(D_s)_{s \geq 1}$  with  $D_1 \subseteq C$ , we have  $D_s \subseteq C_s^{\max}$  for all  $s$ .

The following proposition is a technical ingredient for the main result of this section, Theorem 3.2.

**Proposition 3.1.** *For  $M, N \in \operatorname{Her}_s(\mathbb{C})$ , define*

$$\lambda_1 := \min \left\{ \lambda \in \mathbb{R} \mid \begin{pmatrix} M + \lambda I & N \\ N & I \end{pmatrix} \geq 0 \right\}$$

and

$$\lambda_2 := \min\{\lambda \in \mathbb{R} \mid |w_1|^2 M + 2 \operatorname{Re}(w_1 \overline{w_2}) N + (\lambda |w_1|^2 + |w_2|^2) I \geq 0 \quad \forall w \in \mathbb{C}^2\}.$$

Then  $\lambda_2 \leq \lambda_1$ , and if  $\lambda_2 = \lambda_1$ , then  $M$  and  $N$  have a common eigenvector.

*Proof.* It is well-known that  $\lambda_1 = \max_{\|v\|=1} (v^* N^2 v - v^* M v)$ . Concerning  $\lambda_2$ , it is easy to see that the inequality

$$|w_1|^2 M + 2 \operatorname{Re}(w_1 \overline{w_2}) N + (\lambda |w_1|^2 + |w_2|^2) I \geq 0 \quad \forall w \in \mathbb{C}^2$$

is equivalent to

$$M + 2rN + (\lambda + r^2)I \geq 0 \quad \forall r \in \mathbb{R},$$

and thus to

$$(v^* N v)^2 \leq v^* M v + \lambda \quad \forall \|v\| = 1.$$

Therefore

$$\lambda_2 = \max_{\|v\|=1} [(v^* N v)^2 - v^* M v].$$

We know that  $(v^* N v)^2 = (N v)^* v v^* (N v) \leq (N v)^* I (N v) = v^* N^2 v$  for all  $\|v\| = 1$ , and thus  $\lambda_2 \leq \lambda_1$ . Whenever  $(v^* N v)^2 = v^* N^2 v$ , then  $N v \in \ker(I - v v^*)$ , so  $v$  is an eigenvector of  $N$ . Thus if  $\lambda_2 = \lambda_1$ , then any  $v$  that attains  $\lambda_2$  must also attain  $\lambda_1$ , and therefore be an eigenvector of  $N$ .

We finally show that if  $\lambda_2 = \max_{\|v\|=1} [(v^* N v)^2 - v^* M v]$  is attained at some eigenvector  $v$  of  $N$ , then  $v$  is also an eigenvector of  $M$ . We assume  $\|v\| = 1$  and choose an arbitrary  $w$  with  $\|w\| = 1$  and  $w \perp v$ . Consider the smooth function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(\epsilon) := \left( \frac{(v + \epsilon w)^*}{\|v + \epsilon w\|} N \frac{(v + \epsilon w)}{\|v + \epsilon w\|} \right)^2 - \frac{(v + \epsilon w)^*}{\|v + \epsilon w\|} M \frac{(v + \epsilon w)}{\|v + \epsilon w\|}$$

and compute

$$f'(0) = -w^* M v - v^* M w,$$

where the derivative of the first term vanishes since  $v$  is an eigenvector of  $N$ . Since  $v$  attains  $\lambda_2$ , there is a maximum of  $f$  at  $\epsilon = 0$ , and therefore  $w^* M v + v^* M w = 0$ . This means  $\operatorname{Re}(v^* M w) = 0$ , and by using  $-i w$  instead of  $w$  also  $\operatorname{Im}(v^* M w) = 0$ . Hence  $v^* M w = 0$  for all  $w$  with  $w \perp v$ , which means that the orthogonal complement of  $v$  is invariant under  $M$ . But then  $\mathbb{C}v$  must also be invariant under  $M$ , so that  $v$  is an eigenvector of  $M$ .  $\square$

We can now prove our main result on maximal operator systems:

**Theorem 3.2.** *The operator system  $C^{\max}$  admits a finite-dimensional realization if and only if  $C$  is polyhedral.*

*Proof.* One direction is clear: if  $C = \{a \in \mathbb{R}^d \mid \ell_1(a) \geq 0, \dots, \ell_r(a) \geq 0\}$ , with linear functionals  $\ell_i : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\ell_i(u) = 1$  for all  $i$ , then for all  $s \geq 1$ ,

$$C_s^{\max} = \left\{ A \in \operatorname{Her}_s(\mathbb{C})^d \mid (\ell_1 \otimes \operatorname{id})(A) \geq 0, \dots, (\ell_r \otimes \operatorname{id})(A) \geq 0 \right\},$$

and this gives rise to an  $r$ -dimensional realization with diagonal matrices.

We now show that the maximal system of a non-polyhedral cone does not admit a finite-dimensional realization. First, we argue that we can restrict to the case  $d = 3$ .

Indeed, every non-polyhedral cone  $C$  admits a 3-dimensional linear section through 0 and the order unit  $u$ , which is not polyhedral either [20, Theorem 4.7], and a finite-dimensional realization of  $C^{\max}$  would restrict to a finite-dimensional realization of the maximal system of the section. Moreover, we can assume that  $C$  itself is spectrahedral, since otherwise there is not even a finite-dimensional realization of any system that coincides with  $C$  at scalar level.

Now if  $C \subseteq \mathbb{R}^3$  is non-polyhedral but spectrahedral, then there is an isomorphism  $\varphi \in \text{GL}_3(\mathbb{R})$  such that  $C \cap \varphi(C)$  has nonempty interior, but does not have a face of dimension 2. Indeed, the boundary of  $C$  is an algebraic variety, and so there must be a smooth point with strict curvature. A reflection  $\varphi$  at a suitable hyperplane close to such a point will then work. Since  $C_s^{\max} \cap D_s^{\max} = (C \cap D)_s^{\max}$  holds for any two cones  $C$  and  $D$ , and the intersection of two systems with finite-dimensional realization has a finite-dimensional realization, we can thus assume that  $C$  does not have a face of dimension 2.

Now assume  $M_1, M_2, M_3 \in \text{Her}_r(\mathbb{C})$  are defining matrices for  $C^{\max}$  of minimal matrix size  $r$ . For any  $A = (A_1, A_2, A_3) \in \text{Her}_s(\mathbb{C})^d$ , we then have

$$\begin{aligned} \sum_i M_i \otimes A_i \geq 0 &\Leftrightarrow A \in C_s^{\max} \\ &\Leftrightarrow v^* A v \in C = C_1^{\max} \quad \forall v \in \mathbb{C}^s \\ &\Leftrightarrow \sum_i M_i \cdot v^* A_i v \geq 0 \quad \forall v \in \mathbb{C}^s \\ &\Leftrightarrow \sum_i w^* M_i w \cdot v^* A_i v \geq 0 \quad \forall v \in \mathbb{C}^s, w \in \mathbb{C}^r \\ &\Leftrightarrow \left\langle \left( \sum_i M_i \otimes A_i \right) x, x \right\rangle \geq 0 \quad \forall \text{ elementary tensors } x \in \mathbb{C}^r \otimes \mathbb{C}^s. \end{aligned}$$

Via a suitable change of basis in  $\mathbb{C}^d$ , we can arrive at  $M_3 = I_r$ . The above equivalence then entails that the matrix

$$\begin{pmatrix} M_1 + \lambda I_r & M_2 \\ M_2 & I_r \end{pmatrix}$$

is positive if and only if it is positive on all vectors of the form  $\begin{pmatrix} w_1 v \\ w_2 v \end{pmatrix}$  for  $v \in \mathbb{C}^r$  and  $w = (w_1, w_2) \in \mathbb{C}^2$ . Using Proposition 3.1, it follows that  $M_1$  and  $M_2$ , and trivially also  $M_3$ , have a common eigenvector. Thus we can split off a  $1 \times 1$ -block in each  $M_i$ . Since the corresponding linear inequality is not needed in the linear inequalities description of  $C$  (because there is no face of dimension 2), it is also redundant in the description of the maximal system. This contradicts the minimality of  $r$ .  $\square$

#### 4. THE MINIMAL OPERATOR SYSTEM OF A CONE

Again let  $C \subseteq \mathbb{R}^d$  be a closed salient convex cone with order unit  $u$ . Define

$$C_s^{\min} := \left\{ \sum_i c_i \otimes P_i \mid c_i \in C, P \in \text{Her}_s(\mathbb{C}), P \geq 0 \right\}.$$

**Lemma 4.1.**  $C^{\min}$  is the smallest operator system with  $C_1^{\min} = C$ .

*Proof.* It is clear that  $C^{\min}$  is contained in any operator system extending  $C$ .

It remains to check that each  $C_s^{\min}$  is closed. By Caratheodory's theorem, the number of elementary tensors required to reach every  $A = \sum_i c_i \otimes P_i$  is uniformly bounded. Hence it is enough to show that the set of elementary tensors  $\{c \otimes P : c \in C, P \in \text{Her}_s(\mathbb{C})_+\}$  is closed. By choosing any tensor norm, it follows that the elementary tensors of norm 1 are tensor products of elements of norm 1 and therefore form a compact set.  $\square$

Since it will be a crucial ingredient in our main result, we compute the essential boundary of a particular minimal system:

**Example 4.2.** Consider the cone over the square, i.e.  $C = \text{cc}\{v_1, v_2, v_3, v_4\} \subseteq \mathbb{R}^3$ , where

$$(2) \quad v_1 = (1, -1, 1), \quad v_2 = (-1, 1, 1), \quad v_3 = (1, 1, 1), \quad v_4 = (-1, -1, 1),$$

and  $u = (0, 0, 1)$ . For  $A_1, A_2, A_3, A_4 \geq 0$ , we have

$$v_1 \otimes A_1 + v_2 \otimes A_2 + v_3 \otimes A_3 + v_4 \otimes A_4 \in \partial^{\text{ess}} C_s^{\min}$$

if and only there is some  $U \in \text{GL}_s(\mathbb{C})$  with

$$(3) \quad \text{im}(UA_1) \perp \text{im}(UA_2) \text{ and } \text{im}(UA_3) \perp \text{im}(UA_4).$$

In fact, assume  $\varphi: \text{Her}_s(\mathbb{C})^3 \rightarrow \mathbb{R}$  is nonnegative on  $C_s^{\min}$ . Then

$$\varphi(X, Y, Z) = \text{tr}(XM_1 + YM_2 + ZM_3)$$

for some  $M_1, M_2, M_3 \in \text{Her}_s(\mathbb{C})$  with

$$\pm M_1 \pm M_2 + M_3 \geq 0$$

for all four sign combinations. Furthermore,  $\varphi(u \otimes vv^*) > 0$  for all  $0 \neq v \in \mathbb{C}^s$  just means  $M_3 > 0$ . So there is some  $U \in \text{GL}_s(\mathbb{C})$  with  $(U^{-1})^* M_3 U^{-1} = I_s$ . Now assume

$$\begin{aligned} 0 &= \varphi(v_1 \otimes A_1 + v_2 \otimes A_2 + v_3 \otimes A_3 + v_4 \otimes A_4) \\ &= \text{tr}(A_1(M_1 - M_2 + M_3) + A_2(-M_1 + M_2 + M_3) \\ &\quad + A_3(M_1 + M_2 + M_3) + A_4(-M_1 - M_2 + M_3)). \end{aligned}$$

With  $S = M_1 + M_2$  and  $D = M_1 - M_2$ , the above positivity conditions make this equivalent to

$$A_1 \perp (M_3 + D), \quad A_2 \perp (M_3 - D), \quad A_3 \perp (M_3 + S), \quad A_4 \perp (M_3 - S),$$

where we use the standard inner product  $\langle X, Y \rangle = \text{tr}(Y^* X)$  on matrices. Thus with  $\tilde{D} := (U^{-1})^* D U^{-1}$ ,

$$(4) \quad UA_1 U^* \perp (I_s + \tilde{D}), \quad UA_2 U^* \perp (I_s - \tilde{D}),$$

and similarly for the other two orthogonality relations involving  $\tilde{S} = (U^{-1})^* S U^{-1}$ . Using the spectral decomposition of  $\tilde{D}$  and the fact that eigenvectors to different eigenvalues are orthogonal, we see that  $UA_1 U^*$  and  $UA_2 U^*$  have orthogonal images, and similarly for  $UA_3 U^*$  and  $UA_4 U^*$  with  $S$  in place of  $D$ . This proves (3).



Tracing back this argument, we start with (3), construct  $\tilde{D}$  with spectrum in  $[-1, +1]$  such that (4) holds, and similarly for  $\tilde{S}$ . This determines  $M_1$ ,  $M_2$  and  $M_3$  via the above equations, and all desired properties hold by construction.

Before we can prove our main result of this section, we need some more preliminaries.

**Definition 4.3.**  $C$  has a *universal spectrahedral description* of dimension  $r$  if there are  $M_1, \dots, M_d \in \text{Her}_r(\mathbb{C})$  with

$$\sum_{i=1}^d M_i u_i = I_r, \quad C = \mathcal{S}_1(M_1, \dots, M_d),$$

and whenever  $N_1, \dots, N_d \in \text{Her}_t(\mathbb{C})$  with  $\sum_i N_i u_i = I_t$ , then

$$\mathcal{S}_1(M_1, \dots, M_d) \subseteq \mathcal{S}_1(N_1, \dots, N_d) \Rightarrow \forall s \geq 1 : \mathcal{S}_s(M_1, \dots, M_d) \subseteq \mathcal{S}_s(N_1, \dots, N_d).$$

This means that the representation detects inclusion of free spectrahedra already at scalar level. This is closely related to realizations of minimal operator systems:

**Proposition 4.4.** *Let  $C \subseteq \mathbb{R}^d$  be a closed salient cone. Then the following are equivalent:*

- (i) *The system  $C^{\min}$  is finite-dimensional realizable.*
- (ii)  *$C$  admits a universal spectrahedral description.*

*Proof.* (i) $\Rightarrow$ (ii): Let  $M_1, \dots, M_d$  realize the system. Whenever

$$C = \mathcal{S}_1(M_1, \dots, M_d) \subseteq \mathcal{S}_1(N_1, \dots, N_d),$$

then  $\mathcal{S}_s(M_1, \dots, M_d) = C_s^{\min} \subseteq \mathcal{S}_s(N_1, \dots, N_d)$  for all  $s$ , by minimality.

(ii) $\Rightarrow$ (i): Let  $M_1, \dots, M_d$  be matrices that form a universal spectrahedral description of  $C$ . Then  $C_s^{\min} \subseteq \mathcal{S}_s(M_1, \dots, M_d)$  for all  $s \geq 1$  by minimality. Now assume  $A \notin C_t^{\min}$  for some  $A \in \text{M}_t(\mathbb{C})^d$ . Then by choosing a separating positive functional and applying Lemma 2.1, there are  $N_1, \dots, N_d$  with  $\sum_i N_i u_i = I_t$  and  $C^{\min} \subseteq \mathcal{S}(N_1, \dots, N_d)$ , and such that  $A \notin \mathcal{S}_t(N_1, \dots, N_d)$ . From

$$\mathcal{S}_1(M_1, \dots, M_d) = C = C_1^{\min} \subseteq \mathcal{S}_1(N_1, \dots, N_d),$$

we obtain  $\mathcal{S}(M_1, \dots, M_d) \subseteq \mathcal{S}(N_1, \dots, N_d)$  since the description is universal. Thus  $A \notin \mathcal{S}_t(M_1, \dots, M_d)$ . We have therefore shown  $C_s^{\min} = \mathcal{S}_s(M_1, \dots, M_d)$  for all  $s \geq 1$ .  $\square$

**Lemma 4.5.** *Let  $H \subseteq \mathbb{R}^d$  be a subspace that intersects  $\text{int}(C)$ , and consider the cone  $\tilde{C} := C \cap H$ . Assume that whenever  $\tilde{C} \subseteq \tilde{S} \subseteq H$  for some spectrahedral cone  $\tilde{S}$ , then the matrix pencil defining  $\tilde{S}$  admits an extension to a pencil defining a spectrahedral cone  $S \subseteq \mathbb{R}^d$  containing  $C$ . If  $C^{\min}$  is finite-dimensional realizable, then so is  $\tilde{C}^{\min}$ .*

*Proof.* Assume  $C^{\min}$  is finite-dimensionally realized by  $M_1, \dots, M_d$ . Then the restriction of the pencil  $\text{span}_{\mathbb{R}}\{M_1, \dots, M_d\}$  to  $H$  yields a universal spectrahedral description of  $\tilde{C}$ , by the assumed lifting property. In view of Proposition 4.4,  $\tilde{C}^{\min}$  is finite-dimensional realizable.  $\square$

**Lemma 4.6.** *Let  $\tilde{C}$  be a face of  $C$ . If  $C^{\min}$  is finite-dimensional realizable, then so is  $\tilde{C}^{\min}$ .*

*Proof.* Since  $C$  is in particular spectrahedral,  $\tilde{C}$  must be exposed by some  $\ell \in C^\vee$  [23]. Then for all matrix levels  $s$  and all  $A \in \mathbb{M}_s(\mathbb{C})^d$ ,

$$A \in \tilde{C}_s^{\min} \Leftrightarrow A \in C_s^{\min} \wedge (\ell \otimes \text{id})(A) = 0.$$

Thus every finite-dimensional realization of  $C^{\min}$  restricts to a finite-dimensional realization of  $\tilde{C}^{\min}$ .  $\square$

We are now ready to prove the main result of this section:

**Theorem 4.7.** *For a salient polyhedral cone  $C \subseteq \mathbb{R}^d$ , the system  $C^{\min}$  is finite-dimensional realizable if and only if  $C$  is a simplex. Moreover,  $C^{\min} = C^{\max}$  if and only if  $C$  is a simplex.*

*Proof.* One direction is easy. Any simplex cone is isomorphic to the positive orthant  $C = \mathbb{R}_{\geq 0}^d$ . In this case, one easily checks

$$C_s^{\min} = \left\{ (A_1, \dots, A_d) \in \text{Her}_s(\mathbb{C})^d \mid A_1 \geq 0, \dots, A_d \geq 0 \right\} = C_s^{\max}.$$

We prove the other direction in 3 steps.

*Step 1:* We first deal with the cone over the square  $C = \text{cc}\{v_1, v_2, v_3, v_4\} \subseteq \mathbb{R}^3$  as in Example 4.2, and show that its minimal system is not finite-dimensional realizable. This first nontrivial case is already the hardest. We will use Theorem 2.3 together with our characterization of the essential boundary from Example 4.2. Let

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

be the Pauli matrices. For  $\alpha \in (0, \pi/2)$ , consider the positive semidefinite matrices

$$\begin{aligned} A_1 &= I_2 - \cos(\alpha)\sigma_z + \sin(\alpha)\sigma_x, \\ A_2 &= I_2 + \cos(\alpha)\sigma_z - \sin(\alpha)\sigma_x, \\ A_3 &= I_2 - \cos(\alpha)\sigma_z - \sin(\alpha)\sigma_x, \\ A_4 &= I_2 + \cos(\alpha)\sigma_z + \sin(\alpha)\sigma_x, \end{aligned}$$

where the sign pattern is as in (2), and the associated element

$$(5) \quad A := v_1 \otimes A_1 + v_2 \otimes A_2 + v_3 \otimes A_3 + v_4 \otimes A_4 \in C_2^{\min}.$$

For  $V \in \mathbb{M}_{r,2}(\mathbb{C})$  with columns  $w_1, w_2$ , the property

$$(6) \quad \text{im}(VA_1) \perp \text{im}(VA_2), \quad \text{im}(VA_3) \perp \text{im}(VA_4)$$

just means  $\|w_1\| = \|w_2\|$  and  $w_1 \perp w_2$ . By taking  $V = I_2$  or any other unitary, we conclude  $A \in \partial^{\text{ess}} C_2^{\min}$  by Example 4.2. Now let  $A^{(1)}, \dots, A^{(r)}$  each be as in (5), but for different angles  $0 < \alpha_1 < \dots < \alpha_r < \frac{\pi}{2}$ . If these  $A^{(i)}$  admit a compression to  $\partial^{\text{ess}} C_r^{\min}$  as in Theorem 2.3, then we obtain  $V_i \in \mathbb{M}_{r,2}(\mathbb{C})$  with

$$\begin{aligned} \text{im}(V_1 A_1^{(1)} V_1^* + \dots + V_r A_1^{(r)} V_r^*) &\perp \text{im}(V_1 A_2^{(1)} V_1^* + \dots + V_r A_2^{(r)} V_r^*), \\ \text{im}(V_1 A_3^{(1)} V_1^* + \dots + V_r A_3^{(r)} V_r^*) &\perp \text{im}(V_1 A_4^{(1)} V_1^* + \dots + V_r A_4^{(r)} V_r^*), \end{aligned}$$

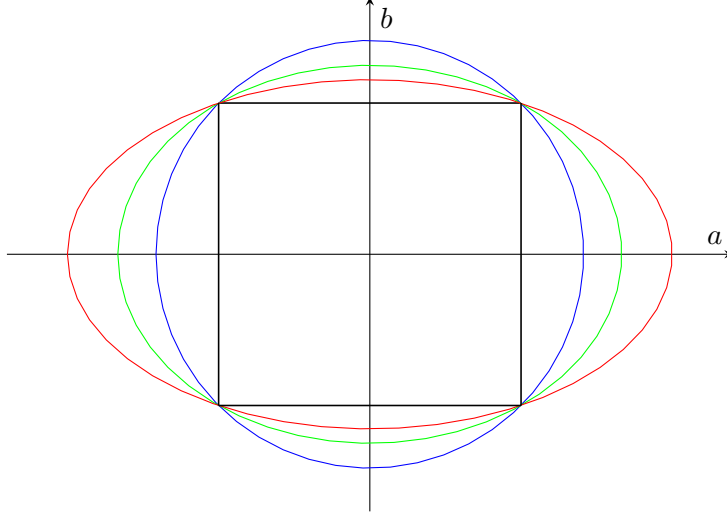


FIGURE 1. A section of the cone  $C$  (square) together with some sections of various  $C(\alpha)$ .

where now the  $U$  of (3) has been absorbed into the  $V_i$ . Since each summand is positive, these orthogonality relations require the individual summands to have orthogonal images,

$$\text{im}(V_i A_1^{(i)} V_i^*) \perp \text{im}(V_j A_2^{(j)} V_j^*), \quad \text{im}(V_i A_3^{(i)} V_i^*) \perp \text{im}(V_j A_4^{(j)} V_j^*),$$

for all  $i, j = 1, \dots, r$ . The  $A_k^{(i)}$  have rank one, and hence so do the  $V_i A_k^{(i)} V_i^*$ . An elementary calculation then shows that the  $2r$  columns of all the  $V_i$ 's must be pairwise orthogonal, which is impossible in a space of dimension  $r$ . Hence  $\sum_i V_i A^{(i)} V_i^*$  cannot be in the essential boundary, and Theorem 2.3 implies that  $C^{\min}$  is not  $r$ -dimensional realizable. So it is not finite-dimensional realizable. This completes Step 1.

*Step 2:* We now generalize to arbitrary salient polyhedral cones in  $\mathbb{R}^3$ . Again let  $C = \text{cc}\{v_1, \dots, v_4\} \subseteq \mathbb{R}^3$  be as in Step 1. For  $\alpha \in (0, \pi/2)$ , consider

$$M_1 := \sin(\alpha)\sigma_z, \quad M_2 := \cos(\alpha)\sigma_x, \quad M_3 := I_2.$$

Then  $C \subseteq C(\alpha) := \{(a, b, c) \in \mathbb{R}^3 \mid aM_1 + bM_2 + cM_3 \geq 0\}$ . Figure 1 shows sections in the plane defined by  $c = 1$  of the cones  $C(\alpha)$  for various values of  $\alpha$ . Again consider  $A \in C_2^{\min}$  as in (5). The functional  $(X, Y, Z) \mapsto \text{tr}(XM_1 + YM_2 + ZM_3)$  even shows that  $A \in \partial^{\text{ess}} D_2^{\min}$  for any convex cone  $C \subseteq D \subseteq C(\alpha)$ . Now assume a convex cone  $D$  fulfills

$$C \subseteq D \subseteq C(\alpha)$$

for *infinitely* many values of  $\alpha \in (0, \pi/2)$ . Then families of  $A^{(i)}$  as above (with different values for  $\alpha$ ) are also in  $\partial D_2^{\min}$ , but cannot be compressed into  $\partial^{\text{ess}} D_r^{\min}$  as in Theorem 2.3, since this would then also work for  $\partial^{\text{ess}} C_r^{\min}$ . Hence the operator system  $D^{\min}$  is not finite-dimensional realizable.

Any quadrilateral in the plane can be transformed with a projective transformation to the square. So in a given planar polytope which is not a simplex, choose

vertices  $u_1, u_2, w_1, w_2$  that form a quadrilateral, such that both pairs  $u_1, u_2$  and  $w_1, w_2$  are adjacent vertices. Then transform them to the square, and choose  $\alpha' > 0$  such that the transformed polytope is contained in  $C(\alpha)$  for all  $0 < \alpha < \alpha'$ . This is possible, since the gradient to  $\det(aM_1 + bM_2 + M_3)$  at  $(a, b) = (1, 1)$  tends to  $(1, 0)$  for  $\alpha \rightarrow 0$ , and similarly at the other three corners of the square. This shows that any non-simplex polyhedral cone in  $\mathbb{R}^3$  is isomorphic to a cone  $D$  with  $C \subseteq D \subseteq C(\alpha)$  for infinitely many values of  $\alpha \in (0, \pi/2)$ . Its minimal system is thus not finite-dimensional realizable. This finishes Step 2.

*Step 3:* We prove the statement in arbitrary dimension  $d \geq 4$  by induction on  $d$ . If  $C$  is not a simplex, then either it has a facet that is not a simplex, or a vertex figure that is not a simplex [26, p. 67]. In the first case we apply the contrapositive of Lemma 4.6, while in the second case we apply Lemma 4.5 to a hyperplane defining the vertex figure. The extension required by Lemma 4.5 is possible by taking the conical hull of  $\tilde{S}$  from the vertex (ray). In both cases we reduce to dimension  $d - 1$ .

Finally, the statement about  $C^{\min} = C^{\max}$  follows from the previous results.  $\square$

*Remark 4.8.* (i) The argument in Step 2 of the previous proof shows that the minimal system of many non-polyhedral cones is not finite-dimensional realizable either. Any cone in  $\mathbb{R}^3$  having a compact section that contains the square and is contained in two different  $C(\alpha)$  is an example.

(ii) The results from [10] provide further evidence that finitely generated operator systems are hardly ever finite-dimensional realizable.

*Remark 4.9.* Every polyhedral cone can be regarded either as the set of positive linear combinations of its finitely many extreme points, or as the set of all points satisfying its finitely many facet inequalities.  $C^{\min}$  extends the first picture to matrix levels, since we take matrix positive combinations of points from  $C$ . On the other hand,  $C^{\max}$  generalizes the second picture, since it is defined by the inequalities of  $C$ . Except for simplices, these two extensions are thus different at matrix level.

**Example 4.10.** There are non-polyhedral cones with a finite-dimensional realizable minimal operator system. One example is the circular cone

$$C = \{(a, b, c) \in \mathbb{R}^3 \mid c \geq 0, a^2 + b^2 \leq c^2\}.$$

It is proven in [12, Corollary 14.15] and [16, Theorem 5.4.10] (which relies mostly on [5, Theorem 7]), that the following linear matrix pencil defines the minimal system:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes z.$$

It is tempting to conjecture that the following pencil  $\mathcal{L}$  defines the minimal system of the analogous cone over the three-dimensional Euclidean ball in  $\mathbb{R}^4$ :

$$\mathcal{L}(x, y, w, z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes y + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes w + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes z,$$

which indeed coincides with that cone at the first matrix level. However, this is not true. It is well-known that there are hermitian  $2 \times 2$ -matrices with the block form

$$X = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

that are positive semidefinite, but cannot be written as  $\sum_i P_i \otimes Q_i$  with positive semidefinite matrices  $P_i, Q_i$ , where all  $P_i$  are of size 2; such matrices are called *entangled* in the language of quantum physics. An easy example is

$$X = \left( \begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right).$$

Now  $0 \leq 2X = \mathcal{L}(A - C, B + B^*, \frac{1}{i}(B - B^*), A + C)$ . If the inequality  $\mathcal{L} \geq 0$  defined the minimal operator system of the cone over the three-dimensional Euclidean ball, then for every  $X$  there would be vectors  $v_i \in C$  and positive semidefinite matrices  $Q_i$  such that

$$\left( A - C, B + B^*, \frac{1}{i}(B - B^*), A + C \right) = \sum_i v_i \otimes Q_i.$$

But then

$$2X = \sum_i \underbrace{\mathcal{L}(v_i)}_{\geq 0} \otimes Q_i,$$

which contradicts the possibility that  $X$  may be entangled.

## 5. INCLUSION OF SPECTRAHEDRA

We explain how our results relate to inclusion testing of spectrahedra. The initial problem is the following:

**Problem 5.1.** Given  $M_1, \dots, M_d \in \text{Her}_r(\mathbb{C})$  and  $N_1, \dots, N_d \in \text{Her}_t(\mathbb{C})$  with

$$\sum_i u_i M_i = I_r, \quad \sum_i u_i N_i = I_t,$$

then is it true that

$$\mathcal{S}_1(M_1, \dots, M_d) \subseteq \mathcal{S}_1(N_1, \dots, N_d)$$

holds in  $\mathbb{R}^d$ ?

Already if  $\mathcal{S}_1(M_1, \dots, M_d)$  is the cone over a  $d$ -dimensional cube, this question arises in interesting applications [3]. In general, it is a hard algorithmic problem (see [19] for an overview and new results). The authors of [3] have thus come up with the following relaxation of the problem:

**Problem 5.2.** Given  $M_1, \dots, M_d \in \text{Her}_r(\mathbb{C})$  and  $N_1, \dots, N_d \in \text{Her}_t(\mathbb{C})$  with

$$\sum_i u_i M_i = I_r, \quad \sum_i u_i N_i = I_t,$$

do there exist  $V_j \in \mathbb{M}_{r,t}(\mathbb{C})$  such that

$$\sum_j V_j^* M_i V_j = N_i$$

for all  $i$ ?

A positive answer to an instance of Problem 5.2 implies a positive answer to the corresponding instance of Problem 5.1. Furthermore, Problem 5.2 can be formulated as a semidefinite feasibility problem, and is thus algorithmically tractable. However, a positive answer to Problem 5.1 does not necessarily imply a positive answer to Problem 5.2. The main result of [13] says that Problem 5.2 is equivalent to  $\mathcal{S}_s(M_1, \dots, M_d) \subseteq \mathcal{S}_s(N_1, \dots, N_d)$  for all  $s \geq 1$ , i.e. to inclusion of the free spectrahedra. This result mostly relies on Choi's characterization of completely positive maps between matrix algebras [5]. Since the inclusion  $\mathcal{S}_1(M_1, \dots, M_d) \subseteq \mathcal{S}_1(N_1, \dots, N_d)$  does not imply the higher inclusions  $\mathcal{S}_s(M_1, \dots, M_d) \subseteq \mathcal{S}_s(N_1, \dots, N_d)$  in general, Problem 5.2 is just a relaxation of Problem 5.1. There exist quantitative measures for exactness of this relaxation [3, 7, 12, 18], which we will explain in more detail below. The first reformulation of our previous result is the following:

**Corollary 5.3.** *Assume  $C = \mathcal{S}_1(M_1, \dots, M_d) \subseteq \mathbb{R}^d$  is a salient polyhedral cone. Then a positive answer to Question 5.1 implies a positive answer to Question 5.2 for all choices of  $N_1, \dots, N_d \in \text{Her}_t(\mathbb{C})$  if and only if  $C$  is a simplex.*

*Proof.* Problems 5.1 and 5.2 are equivalent for all  $N_1, \dots, N_d$  if and only if the  $M_i$  provide a universal spectrahedral description of  $C$ . So the result follows from Proposition 4.4 and Theorem 4.7.  $\square$

*Remark 5.4.* (i) Although the exactness of the relaxation for simplex cones is easy to prove, it seems like it has not been observed in the literature so far. We will use it below to easily obtain error bounds in the non-exact case.

(ii) Corollary 5.3 holds for *any description* of  $C$  by matrices  $M_i$ . So far, only fixed descriptions have been used to deduce error bounds and inexactness results in [3, 7, 12, 18]. It was not clear a priori whether the inexactness could be removed by choosing a better spectrahedral description of the cones. We now see that this is impossible for non-simplex polyhedral cones.

Using our approach, parts of [18, Theorem 4.8] become easy to prove. The result implies that for inclusion of spectrahedra in polyhedra, the relaxation is always exact.

**Proposition 5.5.** *If  $N_1, \dots, N_d$  commute and  $C \subseteq \mathcal{S}_1(N_1, \dots, N_d)$ , then*

$$C^{\max} \subseteq \mathcal{S}(N_1, \dots, N_d).$$

*Proof.*  $P = \mathcal{S}_1(N_1, \dots, N_d)$  is polyhedral, and thus  $P^{\max} = \mathcal{S}(N_1, \dots, N_d)$  by the easy direction in the proof of Theorem 3.2. The claim now follows from  $C^{\max} \subseteq P^{\max}$ .  $\square$

We can also deduce the existence of scaling factors for inclusion, as done in [7, 18]. For a salient convex cone  $C$ , we choose a supporting hyperplane  $H$  that touches  $C$  only at the origin. We then define the scaled cone  $\nu \uparrow C$  by taking the intersection of  $C$  with the affine hyperplane  $u + H$ , scaling the intersection with factor  $\nu$  from the point  $u$ , and taking the conical hull with the origin again. Note that this construction depends on the choice of  $H$  and  $u$ .

**Proposition 5.6.** *Let  $C$  be a closed salient cone. Then for any choice of  $H$  and  $u$  there is some  $\nu > 0$  such that*

$$(7) \quad (\nu \uparrow C)^{\max} \subseteq C^{\min}.$$

*Proof.* After choosing  $H$  and  $u$ , choose  $\nu > 0$  and a simplex cone  $S$  with

$$\nu \uparrow C \subseteq S \subseteq C.$$

We then have

$$(\nu \uparrow C)^{\max} \subseteq S^{\max} = S^{\min} \subseteq C^{\min}. \quad \square$$

For any inclusion of cones  $C \subseteq D$ , we thus also have  $(\nu \uparrow C)^{\max} \subseteq D^{\min}$ . By suitable choice of  $u$ , we can also find a uniform bound on  $\nu$  that only depends on the dimension:

**Theorem 5.7.** *Let  $C \subseteq \mathbb{R}^d$  be a closed salient cone. Then for any choice of  $H$ , there is an order unit  $u \in C$  such that the inclusion  $(\nu \uparrow C)^{\max} \subseteq C^{\min}$  holds with  $\nu = 1/(d+1)$ .*

*Proof.* In the proof of Proposition 5.6, we apply the main theorem of [21]: whenever one inscribes into a convex body in  $\mathbb{R}^{d-1}$  a simplex of maximal volume, then scaling the body with ratio  $1/(d+1)$  from the barycenter of the simplex will make it contained in the simplex.  $\square$

We can also recover the factor of inverse dimension from [7] in the presence of symmetry. Since we talk about cones as opposed to compact convex bodies, this dimension is our  $d-1$ :

**Theorem 5.8.** *Let  $C \subseteq \mathbb{R}^d$  be a closed salient cone, and assume  $C \cap (u+H)$  is symmetric with respect to  $u$ . Then the inclusion (7) holds with  $\nu = 1/(d-1)$ .*

*Proof.* As in the previous proof, one can use the simplex of maximal volume contained in the centrally symmetric convex body  $C \cap (u+H)$ . Grünbaum showed that the scaling factor can then be taken equal to the dimension [11, p. 259]. But since his method would not necessarily yield the center of symmetry  $u$  as the center of scaling, we argue slightly differently.

For notational simplicity, we assume  $H = \mathbb{R}^{d-1} \times \{0\} \cong \mathbb{R}^{d-1}$  and  $u = (0, \dots, 0, 1)$ , which we take to be the origin of  $u+H$  as identified with  $\mathbb{R}^{d-1}$ . Set  $\tilde{C} := C \cap (u+H) \subseteq \mathbb{R}^{d-1}$  and let  $S \subseteq \tilde{C}$  be a simplex of maximal volume. If  $F$  is a face of  $S$  and  $v$  its opposite vertex, then  $F$  supports  $\tilde{C}$ , since otherwise we could increase the volume of the simplex. This is true for any face. So if  $b$  denotes the barycenter of  $S$ , we obtain  $\tilde{C} \subseteq -(d-1)(S-b) + b$ . Symmetry of  $\tilde{C}$  now implies

$$\frac{1}{d-1}\tilde{C} + \frac{d}{d-1}b \subseteq S.$$

The same holds true with  $-b$  and  $-S$  instead. Now assume  $(A_1, \dots, A_d) \in C_s^{\max}$ . Then

$$\left( \frac{1}{d-1}A_1 + \frac{d}{d-1}b_1A_d, \dots, \frac{1}{d-1}A_{d-1} + \frac{d}{d-1}b_{d-1}A_d, A_d \right) \in C_s^{\min},$$

by the argument used for Proposition 5.6. We get the same result with  $-b$  instead of  $b$ , and after adding and dividing by 2, we arrive at the desired conclusion,

$$\left( \frac{1}{d-1}A_1, \dots, \frac{1}{d-1}A_{d-1}, A_d \right) \in C_s^{\min}. \quad \square$$



## REFERENCES

- [1] M. Argerami, S. Coskey, M. Kalantar, M. Kennedy, and M. Lupini, *The classification problem for finitely generated operator systems and spaces*, preprint. [arXiv:1411.0512](#). [↑1](#)
- [2] W. B. Arveson, *Subalgebras of  $C^*$ -algebras*, Acta Math. **123** (1969), 141–224. [↑1](#)
- [3] A. Ben-Tal and A. Nemirovski, *On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty*, SIAM J. Optim. **12** (2002), no. 3, 811–833 (electronic). [↑1](#), [2](#), [13](#), [14](#)
- [4] G. Blekherman, P. A. Parrilo, and R. R. Thomas (eds.), *Semidefinite optimization and convex algebraic geometry*, MOS-SIAM Series on Optimization, vol. 13, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013. [↑1](#), [3](#)
- [5] M. D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra and Appl. **10** (1975), 285–290. [↑12](#), [14](#)
- [6] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Functional Analysis **24** (1977), no. 2, 156–209. [↑2](#)
- [7] K. R. Davidson, A. Dor-On, O. M. Shalit, and B. Solel, *Dilations, inclusions of matrix convex sets, and completely positive maps*, preprint. [arXiv:1601.07993](#). [↑1](#), [14](#), [15](#)
- [8] K. Davidson and M. Kennedy, *The Choquet boundary of an operator system*, Duke Math. J. **164** (2015), no. 15, 2989–3004. [arXiv:1303.3252](#). [↑1](#), [2](#)
- [9] E. G. Effros and S. Winkler, *Matrix convexity: operator analogues of the bipolar and Hahn-Banach theorems*, J. Funct. Anal. **144** (1997), no. 1, 117–152. [↑3](#)
- [10] E. Evert, J.W. Helton, I. Klep, and S. McCullough, *Extreme points of matrix convex sets, free spectrahedra and dilation theory*, in preparation. [↑12](#)
- [11] B. Grünbaum, *Measures of symmetry for convex sets*, Proc. Sympos. Pure Math., 1963, pp. 233–270. [↑15](#)
- [12] J. W. Helton, I. Klep, S. McCullough, and M. Schweighofer, *Dilations, linear matrix inequalities, the matrix cube problem and beta distributions*, Mem. Amer. Math. Soc. [arXiv:1412.1481](#). [↑1](#), [12](#), [14](#)
- [13] J. W. Helton, I. Klep, and S. McCullough, *The matricial relaxation of a linear matrix inequality*, Math. Program. **138** (2013), no. 1-2, Ser. A, 401–445. [arXiv:1003.0908](#). [↑1](#), [14](#)
- [14] J. W. Helton and S. McCullough, *Every convex free basic semi-algebraic set has an LMI representation*, Ann. of Math. (2) **176** (2012), no. 2, 979–1013. [arXiv:0908.4352](#). [↑5](#)
- [15] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, 1948, pp. 187–204. [↑](#)
- [16] K. Kellner, *Positivstellensatz Certificates for Containment of Polyhedra and Spectrahedra*, Doctoral Thesis, University of Frankfurt, Germany. <http://publikationen.uni-frankfurt.de/frontdoor/index/index/docId/37816>. [↑12](#)
- [17] K. Kellner and T. Theobald, *Sum of squares certificates for containment of  $\mathcal{H}$ -polytopes in  $\mathcal{V}$ -polytopes*, SIAM J. Discrete Math. **30** (2016), no. 2, 763–776. [arXiv:1409.5008](#). [↑1](#)
- [18] K. Kellner, T. Theobald, and Ch. Trabant, *A semidefinite hierarchy for containment of spectrahedra*, SIAM J. Optim. **25** (2015), no. 2, 1013–1033. [arXiv:1308.5076](#). [↑1](#), [14](#)
- [19] ———, *Containment problems for polytopes and spectrahedra*, SIAM J. Optim. **23** (2013), no. 2, 1000–1020. [arXiv:1204.4313](#). [↑1](#), [2](#), [13](#)
- [20] V. Klee, *Some characterizations of convex polyhedra*, Acta Math. **102** (1959), 79–107. [↑7](#)
- [21] M. Lassak, *Approximation of convex bodies by inscribed simplices of maximum volume*, Beitr. Algebra Geom. **52** (2011), no. 2, 389–394. [↑15](#)
- [22] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002. [↑2](#)
- [23] M. Ramana and A. J. Goldman, *Some geometric results in semidefinite programming*, J. Global Optim. **7** (1995), no. 1, 33–50. [↑10](#)
- [24] J. F. Sturm, *Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones*, Optimization Methods and Software **11–12** (1999), 625–653. Software available at <https://github.com/sqlp/sedumi>. [↑3](#)



- [25] H. Wolkowicz and R. Saigal and L. Vandenberghe (ed.), *Handbook of semidefinite programming*, International Series in Operations Research & Management Science, 27, Kluwer Academic Publishers, Boston, MA, 2000. Theory, algorithms, and applications. [↑3](#)
- [26] G. M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. [↑12](#)

TOBIAS FRITZ, MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, LEIPZIG, GERMANY

*E-mail address:* `fritz@mis.mpg.de`

TIM NETZER, UNIVERSITÄT INNSBRUCK, AUSTRIA

*E-mail address:* `tim.netzer@math.uibk.ac.at`

ANDREAS THOM, TU DRESDEN, GERMANY

*E-mail address:* `andreas.thom@tu-dresden.de`